The Mathematics of the Spinning Particle Problem *

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Abstract

We introduce an original approach to geometric calculus in which we define derivatives and integrals on functions which depend on extended bodies in space–that is, paths, surfaces, and volumes etc. Though this theory remains to be fully completed, we present it at its current stage of development, and discuss it's connection to physical research, in particular its application to spinning particles in curved space.

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I. Introduction

A. Statement of Purpose

In this paper, we wish to present a generalization of calculus that has been developed in connection with the investigations into the motion of spinning particles in curved space by Pezzaglia[9, 10, 11, 12]. The purpose of this calculus is twofold: First, to provide some mathematical and conceptual support for Pezzaglia's interpretation of his theoretical results. Secondly, to recast the notions of differentiation and integration in a manifestly geometric way, therefore suggesting new possible avenues for pure and applied mathematical research.

The basic idea of this new calculus is quite simple, though its implications are broad: The idea is, simply, to *define calculus* on functions which depend upon extended objects in space–that is, paths, surfaces, volumes etc. Usually, such functions are considered outside the domain of possible consideration in calculus. For example, in Electrostatics, we usually define the scalar potential as the mechanical work per charge:

$$V(\underline{\mathbf{R}}) = \int_P \mathbf{E}(\underline{\mathbf{R}}') \cdot d\mathbf{v}' \ . \tag{1.1}$$

Though this is a path integral, the potential is still only position dependent because $\nabla \wedge \mathbf{E} = 0$ in electrostatics. However, in electro*dynamics* $\nabla \wedge \mathbf{E}$ is not zero, and the work integral eq. (1.1) becomes path dependent. Since we do not understand the calculus of path-dependent functions, we are forced to abandon eq. (1.1) in favor of the "vector potential" concept–even though the work integral seems more physically relevant and intuitively clearer (see, for example, Griffiths[4]). Perhaps, with a more complete calculus of paths and surfaces etc., we won't need to be so theoretically constrained.

In this paper, we would like to trace the evolution of this new calculus from its origins in Clifford Algebra, physics, and multivariable calculus, to its current state of development. At this time, we understand path dependent calculus relatively well, and are still working to find a satisfactory generalization to surfaces and volumes etc. Nevertheless, we have sufficient results to justify this calculus as a possibly interesting avenue for further research, and we hope to encourage more ideas in this direction.

Unfortunately, because of the limited space, we revert to stating our results without proof. A more in-depth version of this paper is available for the interested reader[2].

B. Clifford Algebra and Dimensional Democracy

Many would argue that Clifford Algebra is perhaps the most authentic algebraic representation geometry. Like no other theory, Clifford Algebra integrates all of the dimensions of space into a seamless whole, treating each dimension as equally important. Yet, at the same time, Clifford Algebra has another distinct and powerful asset: it is an extraordinarily useful language for expressing

and understanding the laws of physics, from gravitation to quantum mechanics. One might wonder if this is not a coincidence, but, perhaps, the result of a deep congruence between the content of physical law and the geometric structure of Clifford Algebra. This line of reasoning has lead Pezzaglia[11] to propose a new physical principle, which he calls **Dimensional Democracy:** *The structure behind the laws of physics is reflected in the geometric language of Clifford Algebra. Thus, physical law must itself completely and coherently utilize all of the dimensions of space, as does Clifford Algebra.*

Or, more concisely, *All of the dimensions within space are equally important in determining the laws of physics.* It is important to understand that "Dimensional Democracy" is a *physical principle*, and as such, may be proven incorrect. "Dimensional Democracy" not only asserts that that Clifford Algebra is *useful* in physics, but furthermore asserts that Clifford Algebra's geometric philosophy is *necessary* for a full understanding of physical law.

C. The Spinning Particle Problem

"Dimensional Democracy" is a very broad principle, and can obviously be taken in many directions. Using ideas inspired by Dimensional Democracy, a successful Lagrangian derivation of the Papapetrou equations for the motion of spinning particles in curved space, a long standing problem in classical physics, has been recently solved[12]. In this derivation, Pezzaglia[10] makes use of a particular corollary to "Dimensional Democracy," which can be stated as follows: *In the most general class of physics problems, each dimension within space contributes through its own physical coordinate.*

To understand the motion of spinning particles in curved space, it is necessary to use *both* a position and area coordinate, where the position is conjugate to the particle's momentum and the area is conjugate to its spin. This idea alone leads almost directly to the Papapetrou equations[8, 11, 12]. To see this, consider a particle in the absence of forces,

$$0 = \frac{d}{d\tau} p^\mu \hat{\mathbf{e}}_\mu = \dot{p}^\mu \hat{\mathbf{e}}_\mu + p^\mu \frac{d\hat{\mathbf{e}}_\mu}{d\tau} \ . \tag{1.2}$$

If the particle had no spin, we would derive the geodesic equation as usual by invoking the chain rule,

$$\frac{d\hat{\mathbf{e}}_\mu}{d\tau} = \frac{dx^\nu}{d\tau} \frac{d\hat{\mathbf{e}}_\mu}{dx^\nu} = \dot{x}^\nu \Gamma_{\mu\nu}^{\ \ \sigma} \hat{\mathbf{e}}_\sigma \ . \tag{1.3}$$

However, this is not correct for a spinning particle, because now we have area coordinates as well as position coordinates; therefore we need a more complete chain rule, one which includes derivatives with respect to the area as well as position:

$$\frac{d\hat{\mathbf{e}}_\mu}{d\tau} = \frac{dx^\nu}{d\tau} \frac{\partial \hat{\mathbf{e}}_\mu}{\partial x^\nu} + \frac{1}{2} \frac{da^{\sigma\delta}}{d\tau} \frac{\partial \hat{\mathbf{e}}_\mu}{\partial a^{\sigma\delta}} = \dot{x}^\nu \Gamma_{\mu\nu}^{\ \ \sigma} \hat{\mathbf{e}}_\sigma + \frac{1}{2} \dot{a}^{\sigma\delta} \frac{\partial \hat{\mathbf{e}}_\mu}{\partial a^{\sigma\delta}} \ . \tag{1.4}$$

This is interesting, but what does it mean to take a derivative with respect to area? As it turns out, to derive the Papapetrou equations, it is necessary to propose that the area derivative is actually a commutator of two position derivatives[9]:

$$\frac{\partial}{\partial a^{\mu\nu}} \equiv \frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial x^{\nu}} - \frac{\partial}{\partial x^{\nu}}\frac{\partial}{\partial x^{\mu}} = \left[\frac{\partial}{\partial x^{\mu}}, \frac{\partial}{\partial x^{\nu}}\right] \ . \tag{1.5}$$

With this definition, the area derivative of the basis vector becomes a curvature tensor, and we find:

$$0 = \left(\dot{p}^{\mu} + p^{\sigma}\dot{x}^{\delta}\Gamma_{\sigma\delta}^{\ \ \mu} + \frac{1}{2}\dot{x}^{\gamma}S^{\sigma\delta}R_{\sigma\delta\gamma}^{\ \ \ \ \mu}\right)\hat{\mathbf{e}}_{\mu} \ . \tag{1.6}$$

This is one of the Papapetrou equations[8]. One's reaction may be mixed. This derivation only supports the notion of "Dimensional Democracy" insofar as it is, somehow, geometrically *appropriate* to define the area derivative as a commutator derivative. A primary motivation for this paper is to affirm that, indeed, this definition is very appropriate, provided one views calculus in a properly geometric way.

II. Review of Position-Dependent Calculus

For the purpose of developing a more suggestive and appropriate notation, we begin by re-deriving and re-expressing familiar results from multivariable calculus. Thus, the next few sections will not be new, but will establish basic themes which will continue to guide us as we consider more complicated types of functions later.

Multivariable calculus is concerned with functions which map a *point* in an n-dimensional Euclidean space to an element from a linear vector space, V. The exact nature of V will not be important for us here, except that addition and multiplication by scalars must be defined on it. Also, because we would like to construct *vectors* within our Euclidean space, we imagine that each axis in the space is supplied with a basis vector. Thus, we can represent the location of any point by using a position vector $\underline{\mathbf{R}}$. (The underline signifies that the vector is to be interpreted as a displacement with respect to the origin –that is, *not* a free vector.) So, in standard notation, we are considering all functions $F(\underline{\mathbf{R}})$ such that $F : E^n \rightarrow V; \underline{\mathbf{R}} \mapsto F(\underline{\mathbf{R}})$, where E^n is the set of all Clifford vectors in the n-D Euclidean space (as in Baylis[1]).

A. The Displacement Derivative

We now seek some way to quantify the *change* in F as a function of $\underline{\mathbf{R}}$. To this end, we define the *displacement derivative*, $\mathbf{d_v} : F \rightarrow V; F(\underline{\mathbf{R}}) \rightarrow \mathbf{d_v}F(\underline{\mathbf{R}})$,

$$d_{\mathbf{v}}F(\underline{\mathbf{R}}) \equiv \epsilon \lim_{\alpha\rightarrow 0} \alpha^{-1}\Big(F(\underline{\mathbf{R}} + \alpha\mathbf{v}) - F(\underline{\mathbf{R}})\Big) \ . \tag{2.1}$$

In the above equation, "ϵ" is an extremely small quantity-an infinitesimal. Just stating this will be sufficient for our purposes, but the interested reader will note that our treatment of infinitesimals is essentially in keeping with the notions of Non-Standard Analysis[13].

The reason why we care about infinitesimals like "ϵ" is that they allow a very simple interpretation of the displacement derivative, $\mathbf{d_v}$. To see this, we note,

$$F(\underline{\mathbf{R}} + \epsilon\mathbf{v}) - F(\underline{\mathbf{R}}) = d_{\mathbf{v}}F(\underline{\mathbf{R}}) + O(F\epsilon^2|\mathbf{v}|) \ . \tag{2.2}$$

Thus, $\mathbf{d_v}$ computes the leading infinitesimal change in $F(\underline{\mathbf{R}})$ that results from a slight change of $\underline{\mathbf{R}}$ in the direction of $\mathbf{v}$. The displacement derivative $\mathbf{d_v}$ has one other particularly important feature: it is *linear* in its vector argument. This property is called "internal linearity", which we express by the equation,

$$\mathbf{d_{v+w}} = \mathbf{d_v} + \mathbf{d_w} \ , \tag{2.3a}$$

$$\mathbf{d}_{c\mathbf{v}} = c\,\mathbf{d_v} \ , \ c \in \Re \ . \tag{2.3b}$$

Internal linearity reveals the most crucial insight (and assumption) of calculus; that locally, functions appear *linear*.

B. Integrals and the Fundamental Theorem

We now define the path integral, a higher-dimensional analogy to the usual $\int f(x)dx$ in single variable calculus. Mainly what's new in n-dimensions is that we are no longer constrained to integrate along a straight line, but are free to move about in space while we integrate. So, now, we integrate along a curved path described by the parametric equation,

$$\underline{\mathbf{R}} = \underline{\mathbf{R}}(t) \ , \ \text{For} \ \ t \in [0, T] \ . \tag{2.4}$$

Given such a path, we can construct a "path integral operator", which we symbolize as $\int_{\underline{\mathbf{R}}(t);t\in[0,T]}$. This operator is defined on a function, $K(\underline{\mathbf{R}}, \mathbf{v})$, which depends both on position, $\underline{\mathbf{R}}$ and the rate of change in position, $\mathbf{v}$. Thus, $K : E^n \times E^n \to V; (\underline{\mathbf{R}}, \mathbf{v}) \mapsto K(\underline{\mathbf{R}}, \mathbf{v})$. We define the path integral of K as a mapping,

$$\int_{\underline{\mathbf{R}}(t);t\in[0,T]} : V \to V; K(\underline{\mathbf{R}}, \mathbf{v}) \mapsto \int_{\underline{\mathbf{R}}(t);t\in[0,T]} K(\underline{\mathbf{R}}', \mathbf{v}') \ , \tag{2.5}$$

which can be computed by the Reimann sum:

$$\int_{\underline{\mathbf{R}}(t);t\in[0,T]} K(\underline{\mathbf{R}}', \mathbf{v}') \equiv \frac{1}{\epsilon} \lim_{NT\to\infty} N^{-1} \sum_{n=0}^{T/\alpha} K\left(\underline{\mathbf{R}}(n\alpha), \left.\frac{d\underline{\mathbf{R}}}{dt}\right|_{t=n\alpha}\right) \ . \tag{2.6}$$

Now that we've defined both the displacement derivative and the path integral by equations (2.1) and (2.6), we combine the definitions to form the fundamental theorem of calculus:

$$\int_{\underline{\mathbf{R}}(t); \ t\in[0,T]} \mathbf{d_v} F(\underline{\mathbf{R}}') = F[\underline{\mathbf{R}}(T)] - F[\underline{\mathbf{R}}(0)] \ . \tag{2.7}$$

Of course, there are other fundamental theorems in n-dimensions–the so-called Stoke's theorems, which integrate over surfaces, volumes etc. Hestenes[5] showed that we could write all such fundamental theorems in the form:

$$\int_{X^k} \left(d\mathbf{x}'^k \cdot \nabla\right) F(\underline{\mathbf{R}}') = \oint_{\delta X^k} d\mathbf{x}^{k-1} F(\underline{\mathbf{R}}') \ , \tag{2.8}$$

where $\int_{X^k}$ indicates integration over some k-dimensional body and $\oint_{\delta X^k}$ indicates integration around the surface of that body. For the most part, we will be concerned only with the path integral fundamental theorem, except to note one thing: all fundamental theorems as in eq. (2.8) integrate in steps of one. That is, it relates an integral over a k-surface to an integral over a k-1 surface, and the integral of a single derivative ∇ to an integral with no derivatives. As we consider path and surface dependent functions, we will find fundamental theorems which, amazingly, do not integrate in steps of one!

C. Hints of a Path Dependent Calculus

In single variable calculus, the fundamental theorem usually comes in two distinct forms, one with the derivative *inside* the integral and one with it *outside*:

$$\int_{x_0}^{x} \frac{df}{dx'} dx' \equiv f(x) - f(x_0) \ , \qquad \frac{d}{dx} \int_{x_0}^{x} f(x')dx' \equiv f(x) \ . \tag{2.9ab}$$

The first equation (2.9a) is basically the path integral theorem, eq. (2.7), expressed in one dimension. The second one (2.9b), on the other hand, doesn't appear to have any analogue in higher dimensions, at least not that we've discussed. We could extrapolate this equation, however; perhaps the generalization could look something like,

$$\mathbf{d_v} \int_{\underline{\mathbf{R}}(t);t\in[0,T]} F(\underline{\mathbf{R}}') = F[\underline{\mathbf{R}}(T)] \ . \tag{2.10}$$

However, a moment's thought reveals that this apparently natural equation has no meaning! The displacement derivative is defined on functions of position, *not* on path dependent functions like $\int F$, so eq. (2.10) is effectively meaningless. For this reason, mathematicians have usually presumed that there is no generalization of the fundamental theorem with the derivative *outside* the integral. However, this is not the only line we could take on this issue, perhaps we should, instead, *define* differentiation on path dependent objects so that theorems like eq. (2.10) make sense.

Further indication that we might need a path dependent calculus comes from the Pezzaglia's "area" derivative, eq. (1.5). It is simple to show that displacement derivatives always commute when they act on position dependent functions:

$$[\mathbf{d_v}, \mathbf{d_w}]\, F(\underline{\mathbf{R}}) = 0 \ . \tag{2.11}$$

Unfortunately, this is bad news for the area derivative, because eq. (1.5) defines it to be precisely the commutator derivative, which is always zero! Clearly

something more is needed, and there is reason to think that a path-dependent calculus might solve this problem. In particular, consider the following: The displacement derivative $\mathbf{d_v}$ describes the change in F through a displacement which is bounded by two points. If the action of $\mathbf{d_v}$ is to be nonzero, clearly the function must depend on position, otherwise the function would not be able to detect the change between endpoints. By analogy, perhaps the area derivative describes the change in F through an *area which is bounded by two paths.* Therefore, F must be at least path dependent for a nontrivial area derivative.

III. Derivatives in Path Dependent Calculus

We devote the next few sections to extending the definition of the derivative to path dependent functions. The concepts and notation are slightly more involved here, but the added complication is compensated for by increased mathematical richness, which has yet to be fully explored. In part IV we will define integration on path dependent functions as well, from which we will be able to derive new and interesting fundamental theorems.

A. Representing a Path

This section is devoted to developing a useful notation for representing paths as mathematical objects. We could, of course, describe paths in the normal way by thinking of them as parameterized sets of points as described in eq. (2.4). Unfortunately, this representation is too difficult to work with for our purposes. So, let us propose a more practical notation based on an operation called *stacking*. Stacking, symbolized by $\Rightarrow$, is basically the process of bringing two objects together to form an ordered pair. Thus, $a \Rightarrow b = (a, b), \quad \text{for } a, b \in S$, where "$S$" is some arbitrary set. Really, stacking is no more than a new symbolism for designating ordered pairs, but it portrays ordered pairs in a more dynamic fashion, as an *operation* which is performed between two objects, rather than as a completed object in itself. Any object which is created by stacking other objects together is called a *stack*. Notice that stacking is *not* communitive; this property will be what allows us to define non-commuting derivatives, as we will soon see.

Of particular interest to us is are stacks composed of *vectors* in E^n, for example,

$$s = \mathbf{u}_1 \Rightarrow \mathbf{u}_2 \Rightarrow \ldots \Rightarrow \mathbf{u}_n \ . \tag{3.1}$$

For the purpose of visualization, we imagine that stacks such as eq. (3.1) look like sequences of vectors which have been "glued" on top of one another, head to tail, like a structure of rigid rods in space. The motivation behind this visualization is that we can create a continuous path by simply stacking together an infinite sequence of infinitesimal vectors. Thus, if we are given a parametric equation

$\underline{\mathbf{R}} = \underline{\mathbf{R}}(t), t \in [0, T]$, we can write a path,

$$P \equiv \lim_{NT\to\infty} \sum_{n=1}^{NT} \Rightarrow [\underline{\mathbf{R}}(n/N) - \underline{\mathbf{R}}([n-1]/N)] \ . \tag{3.2}$$

In the above equation, the $\sum \Rightarrow$ symbol simply denotes a "summation" carried out with arrows rather than pluses, (which, incidentally, means we have to be careful to carry out the "sum" in the specified order). As eq. (3.2) is written, the path P doesn't have any particular location within space, but we can define its location by adjoining it with a position vector, $\underline{P} = \underline{\mathbf{R}}(0) \Rightarrow P$, which fixes the starting end of the path to the position $\underline{\mathbf{R}}(0)$. Given a path $\underline{\mathrm{P}}$, we would like to define an operation that allows us to break it into smaller pieces. Thus, we define the "partition" of a path, symbolized ${}_a(P)_b$,

$$_a(P)_b \equiv \lim_{NT\to\infty} \sum_{n=Na}^{Nb} \Rightarrow [\underline{\mathbf{R}}(n/N) - \underline{\mathbf{R}}([n-1]/N)] \ . \tag{3.3}$$

We assume that $0 \leq a \leq b \leq T$. In the special cases $a = 0$ or $b = T$, we will simply write ${}_0(P)_b = (P)_b$ and ${}_a(P)_T = {}_a(P)$.

To shorten our equations later, we symbolize a straight line path in the direction of a vector $\mathbf{v}$ by,

$$L_T\mathbf{v} = \lim_{NT\to\infty} \sum_{n=1}^{NT} \Rightarrow (n/N)\mathbf{v} \ . \tag{3.4}$$

B. The Displacement Derivative

We now extend the definition of a the displacement derivative so that it can act on *path dependent* functions. By "path dependent function", we mean any mapping, G, which takes a path $\underline{P}$ into an element of a linear vector space V. Thus, denoting the set of all paths in E^n by Φ^n, we can define G as a function, $G : \Phi^n \to V;\ \underline{P} \mapsto G(\underline{P})$. Notice that we have underlined the path $\underline{P}$ to remind us that its location in space has been fixed.

Now, when we take the derivative of a path dependent function, we do not vary the path in an arbitrary way, rather we choose a particular point along the path we want to vary, and then change the path at this point *only*. This may seem restrictive, but if we want to compute a more general variation, we can "add up" derivatives taken at different points along the path. So, before we can define the derivative, we have to invent a notation that will tell us which point along the path $\underline{P}$ we are allowed to change as we differentiate. We signify this by placing an asterisk '$\star$' at the chosen point along the path, like so: $\underline{P} = \underline{P}_i^\star \Rightarrow P_f$. Here $\underline{P}_i$ is the part of the path preceding the point of differentiation (in the stacking sense), and P_f is the part after.

We are now ready to consider the displacement derivative, which we define as the mapping, $\mathbf{d_v} : V \to V;\ G(\underline{P}) \mapsto \mathbf{d_v}G(\underline{P})$ such that,

$$\mathbf{d_v}G(\underline{P}_i^\star \Rightarrow P_f) \equiv \epsilon \lim_{\alpha\to 0} \alpha^{-1} \left[G(\underline{P}_i^\star \Rightarrow L_\alpha\mathbf{v} \Rightarrow P_f) - G(\underline{P}_i^\star \Rightarrow P_f)\right] \ , \tag{3.5}$$

where $L_\alpha \mathbf{v}$ is a straight line path in the direction of $\mathbf{v}$, as in eq. (3.4). By making a Taylor expansion, we can see that this definition has a simple interpretation:

$$G\left(\underline{P}_i^\star \Rightarrow L_\epsilon \mathbf{v} \Rightarrow P_f\right) - G(\underline{P}_i^\star \Rightarrow P_f) = \mathbf{d_v} G(\underline{P}_i^\star \Rightarrow P_f) + O(\epsilon^2 G|\mathbf{v}|) \ . \quad (3.6)$$

Thus, $\mathbf{d_v}$ computes the first order change in G that results when we insert an infinitesimal path $L_\epsilon \mathbf{v}$ between the paths $\underline{P}_i$ and P_f.

Before we can accept eq. (3.5) as an acceptable derivative, we must demonstrate that $\mathbf{d_v}$ satisfies external linearity and obeys a Leibniz product rule. The external linearity of $\mathbf{d_v}$ is trivial, and a product rule is easily derived:

$$\mathbf{d_v} c(\underline{P}) G(\underline{P}) = c(\underline{P}) \ \mathbf{d_v} G(\underline{P}) + G(\underline{P}) \ \mathbf{d_v} c(\underline{P}) \ , \quad (3.7)$$

for $c \in \Re$; $G \in V$. Therefore, eq. (3.5) defines an acceptable derivative operator.

The question remains as to whether the displacement derivative satisfies internal linearity, eq. (2.3), when it acts on path dependent functions (that is, whether path dependent functions are geometric). Unlike the position dependent case, we cannot provide any reasonably general proof of internal linearity for path dependent functions. Therefore, we restrict ourselves from the outset to considering only path dependent functions which are geometric.

This is great, but are there any geometric path dependent functions? Indeed, there are. For example, the line integral,

$$G(\underline{P}) = \int_{\underline{P}} A(\underline{\mathbf{R}}') d\mathbf{v} \ , \quad (3.8)$$

allows internal linearity to be satisfied (assuming that $A(\underline{\mathbf{R}})$ is geometric). Is this the only example? Thankfully, no, because we can use eq. (3.8) as the "seed" from which we can create an infinite variety of new geometric functions. In particular, we can take any number of line integrals, combine them using any smoothly varying function we like, and the result will still be geometric. Clearly, we could create a huge number of different functions this way, so we can see that internal linearity is not too stringent a condition to impose.

C. The Area Derivative

As we saw earlier, there is strong reason to think that path dependent calculus may possess the key to understanding the area derivative, and also to providing some support to Pezzaglia's geometric derivation of the Papapetrou equations[11, 12]. In this section, we see how far we can go towards realizing these goals. Thus, we define the area derivative $\mathbf{d}^2_{\mathbf{v}\wedge\mathbf{w}}$ as a mapping, $\mathbf{d}^2_{\mathbf{v}\wedge\mathbf{w}} : V \to V; G(\underline{P}) \mapsto \mathbf{d}^2_{\mathbf{v}\wedge\mathbf{w}} G(\underline{P})$, given by the limit,

$$\mathbf{d}^2_{\mathbf{v}\wedge\mathbf{w}} \, G(\underline{P}_i^\star \Rightarrow P_f) \equiv$$

$$\epsilon^2 \lim_{\alpha \to 0} \alpha^{-2} \Big[G\left(\underline{P}_i^\star \Rightarrow L_\alpha \mathbf{v} \Rightarrow L_\alpha \mathbf{w} \Rightarrow P_f\right) - G\left(\underline{P}_i^\star \Rightarrow L_\alpha \mathbf{w} \Rightarrow L_\alpha \mathbf{v} \Rightarrow P_f\right) \Big]. \quad (3.9)$$

This definition may strike the reader as dubious, because both vectors $\mathbf{v}$ and $\mathbf{w}$ are individually needed to compute the limit, whereas $\mathbf{d}^2_{\mathbf{v}\wedge\mathbf{w}}$ should depend only on the area $\mathbf{v}\wedge\mathbf{w}$, by hypothesis. We will discuss this problem in a moment, but for now let's see how to interpret eq. (3.9). By making a Taylor expansion, we can derive:

$$G(\underline{P}_i^\star \Rightarrow L_\epsilon\mathbf{v} \Rightarrow L_\epsilon\mathbf{w} \Rightarrow P_f) - G(\underline{P}_i^\star \Rightarrow L_\epsilon\mathbf{w} \Rightarrow L_\epsilon\mathbf{v} \Rightarrow P_f) = \mathbf{d}^2_{\mathbf{v}\wedge\mathbf{w}}G(\underline{P}) + O(\epsilon^2 G|\mathbf{v}||\mathbf{w}|). \tag{3.10}$$

This tells us that the area derivative computes the lowest order change in G that occurs between two paths that enclose a little parallelogram of area $\epsilon^2\mathbf{v}\wedge\mathbf{w}$. Thus, the area derivative encapsulates the idea of change through a small area, just as the displacement derivative describes change through an infinitesimal displacement.

If the proposition eq. (1.5) is correct, it would seem that we should be able to write our area derivative eq. (3.9) as the commutator of two displacement derivatives. In fact, it is simple to show that we can:

$$\mathbf{d}^2_{\mathbf{v}\wedge\mathbf{w}} = \mathbf{d}_\mathbf{v}\mathbf{d}_\mathbf{w} - \mathbf{d}_\mathbf{w}\mathbf{d}_\mathbf{v} = [\mathbf{d}_\mathbf{v}, \mathbf{d}_\mathbf{w}] \ . \tag{3.11}$$

Thus, we've succeeded at explaining why the derivative with respect to area should be a commutator derivative. For the purpose touching base with standard theory, we re-express eq. (3.11) using the familiar notation of tensors and covariant derivatives:

$$[\nabla_\mu, \nabla_\nu]\, V^\alpha = R_{\mu\nu\beta}{}^\alpha\, V^\beta \ . \tag{3.12}$$

This is a well-known result from differential geometry (without torsion), and can be derived straightforwardly from eq. (3.11). We note, however, that eq. (3.11) is actually more general than eq. (3.12), because it applies to every geometric path dependent function. Equation (3.12), on the other hand, only applies to vector functions in curved space.

We now consider the pressing question of whether or not the area derivative, as given by the limit eq. (3.9), is really a well defined object. To see why we should be concerned, consider the two area derivatives: $\mathbf{d}^2_{\mathbf{v}\wedge\mathbf{w}}G(\underline{P})$ and $\mathbf{d}^2_{[\mathbf{v}+a\mathbf{w}]\wedge[\mathbf{w}+b(\mathbf{v}+a\mathbf{w})]}G(\underline{P})$. It is clear that these two derivatives should be the same, because they differentiate with respect to the same area:

$$[\mathbf{v}+a\mathbf{w}]\wedge[\mathbf{w}+b(\mathbf{v}+a\mathbf{w})] = [\mathbf{v}+a\mathbf{w}]\wedge\mathbf{w} + b[\mathbf{v}+a\mathbf{w}]\wedge[\mathbf{v}+a\mathbf{w}] = \mathbf{v}\wedge\mathbf{w} + a\mathbf{w}\wedge\mathbf{w} = \mathbf{v}\wedge\mathbf{w} \ . \tag{3.13}$$

However, if we take definition eq. (3.9) literally, we should compute $\mathbf{d}^2_{\mathbf{v}\wedge\mathbf{w}}$ and $\mathbf{d}^2_{[\mathbf{v}+a\mathbf{w}]\wedge[\mathbf{w}+b(\mathbf{v}+a\mathbf{w})]}$ using two completely different limits. Unless by some miracle these limits happen to converge to the same value, we must abandon the area derivative concept to avoid contradiction. In fact, for a general path dependent function, these two limits will be different, and we cannot define the area derivative. However, if the function is *geometric*, they will not be different. The easiest way to demonstrate this is to rewrite the limits as commutator

derivatives, invoke internal linearity, and sort out the terms. Therefore, the area derivative is well defined as long as it acts on a geometric function.

There are two things we should check before we accept eq. (3.9) as an acceptable derivative: that it acts linearly on functions and obeys the Leibniz product rule. The area derivative is clearly linear,

$$\mathbf{d}^2_{\mathbf{v}\wedge\mathbf{w}}[G(\underline{P}) + H(\underline{P})] = \mathbf{d}^2_{\mathbf{v}\wedge\mathbf{w}}G(\underline{P}) + \mathbf{d}^2_{\mathbf{v}\wedge\mathbf{w}}H(\underline{P}), \tag{3.14a}$$

$$\mathbf{d}^2_{\mathbf{v}\wedge\mathbf{w}}\, c\, G(\underline{P}) = c\, \mathbf{d}^2_{\mathbf{v}\wedge\mathbf{w}}G(\underline{P}), \tag{3.14b}$$

for $c \in \Re$ and $G, H \in V$. The product rule follows directly from the area-commutator derivative relation eq. (3.11),

$$\mathbf{d}^2_{\mathbf{v}\wedge\mathbf{w}}c(\underline{P})\, G(\underline{P}) = c(\underline{P})\, \mathbf{d}^2_{\mathbf{v}\wedge\mathbf{w}}G(\underline{P}) + G(\underline{P})\, \mathbf{d}^2_{\mathbf{v}\wedge\mathbf{w}}c(\underline{P}) \ , \tag{3.15}$$

for $c \in \Re$ and $G \in V$. Therefore, the area derivative is an acceptable differential operator.

We hope that the reader has been convinced that eq. (3.9) describes what should rightly be called the "area derivative", not only because it is consistent with the area derivative of eq. (1.5), but because it seems to be the uniquely natural extension of the derivative concept from a 1-D "displacement" to a 2-D "area".

D. The Volume Derivative

The reader is doubtless familiar with the fact that the derivative of a constant is zero,

$$\mathbf{d}_{\mathbf{v}}c = 0 \ . \tag{3.16}$$

We have also seen that the area derivative of a position dependent function is always zero,

$$\mathbf{d}^2_{\mathbf{v}\wedge\mathbf{w}}F(\underline{\mathbf{R}}) = 0 \ . \tag{3.17}$$

So, by analogy, we might think that there is a "volume" derivative, $\mathbf{d}^3_{\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}}$ which gives zero when it acts on a path dependent function:

$$\mathbf{d}^3_{\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}}G(\underline{P}) = 0 \ . \tag{3.18}$$

Equations (3.16-18) show an interesting geometric progression, but we need to define the volume derivative before this trend can be concretely realized.

So, let's imagine how we might quantify the change in a path dependent function with respect to a volume. This attempt may seem suspect, because changing a path will sweep out a surface, not a volume. However, perhaps we can have a path sweep out a closed surface which *bounds* a volume, and define the resulting change to be the volume derivative. This means, specifically, adding up the changes which occur as we move the path across each face of the parallelepiped of volume $\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}$:

$$\begin{aligned}\mathbf{d}^3_{\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}}G(\underline{P}) \equiv \quad & \text{(change across right side)} & &+\text{(change across top)}\\ &+\text{(change across front side)} & &+\text{(change across left side)}\\ &+\text{(change across bottom)} & &+\text{(change across back side)}\end{aligned}$$

or more precisely,

$$\mathbf{d}^3_{\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}} G(\underline{P}) \equiv$$

$$\epsilon^3 \lim_{\alpha\to 0} \alpha^{-3} \begin{pmatrix} G\left(\left(\mathbf{w},\mathbf{u}\right),\mathbf{v}\right) - G\left(\left(\mathbf{u},\mathbf{w}\right),\mathbf{v}\right) + G\left(\mathbf{w},\left(\mathbf{v},\mathbf{u}\right)\right) - G\left(\mathbf{w},\left(\mathbf{u},\mathbf{v}\right)\right) \\ +G\left(\left(\mathbf{v},\mathbf{w}\right),\mathbf{u}\right) - G\left(\left(\mathbf{w},\mathbf{v}\right),\mathbf{u}\right) + G\left(\mathbf{v},\left(\mathbf{u},\mathbf{w}\right)\right) - G\left(\mathbf{v},\left(\mathbf{w},\mathbf{u}\right)\right) \\ +G\left(\left(\mathbf{u},\mathbf{v}\right),\mathbf{w}\right) - G\left(\left(\mathbf{v},\mathbf{u}\right),\mathbf{w}\right) + G\left(\mathbf{u},\left(\mathbf{w},\mathbf{v}\right)\right) - G\left(\mathbf{u},\left(\mathbf{v},\mathbf{w}\right)\right) \end{pmatrix} \tag{3.19a}$$

$$\text{where}: G\left(\left(\mathbf{x},\mathbf{y}\right),\mathbf{z}\right) \equiv G[\underline{P}_i \Rightarrow (L_\alpha \mathbf{x} \Rightarrow L_\alpha \mathbf{y}) \Rightarrow L_\alpha \mathbf{z} \Rightarrow P_f] \ , \tag{3.19b}$$

$$\text{and} \quad : G\left(\mathbf{x},\left(\mathbf{y},\mathbf{z}\right)\right) \equiv G[\underline{P}_i \Rightarrow L_\alpha \mathbf{x} \Rightarrow (L_\alpha \mathbf{y} \Rightarrow L_\alpha \mathbf{z}) \Rightarrow P_f] \ . \tag{3.19c}$$

At a glance, we realize that all these terms add to precisely zero. Thus it seems that, though we can move a path around a volume in our imagination, path dependent functions won't know about it! To get a nontrivial volume derivative, we have to step beyond path dependent functions and consider *surface* dependence.

It is straightforward to show that each term in eq. (3.19) can be reproduced by computing,

$$\mathbf{d}^3_{\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}} = \left[\mathbf{d}^2_{\mathbf{u}\wedge\mathbf{w}}, \mathbf{d}_{\mathbf{w}}\right] + \left[\mathbf{d}^2_{\mathbf{v}\wedge\mathbf{w}}, \mathbf{d}_{\mathbf{u}}\right] + \left[\mathbf{d}^2_{\mathbf{w}\wedge\mathbf{u}}, \mathbf{d}_{\mathbf{v}}\right] \ . \tag{3.20}$$

Surprisingly, this equation directly implies that *displacement derivatives don't associate on surface dependent functions.* When we speak of "non-associativity", we do not mean it in the usual sense of ordering operations. In other words, $(\mathbf{d}_{\mathbf{u}}\mathbf{d}_{\underline{\mathrm{V}}})\mathbf{d}_{\mathbf{w}}$ does not mean "first take $\mathbf{d}_{\mathbf{u}}\mathbf{d}_{\mathbf{v}}$ and then multiply $\mathbf{d}_{\mathbf{w}}$ from the left"; this doesn't really make sense in the context of differentiation. What we mean is that, for any string of three derivatives, $\mathbf{d}_{\mathbf{u}}\mathbf{d}_{\mathbf{v}}\mathbf{d}_{\mathbf{w}}$, two of them must be "taken together", in some appropriate sense, in order for $\mathbf{d}_{\mathbf{u}}\mathbf{d}_{\mathbf{v}}\mathbf{d}_{\mathbf{w}}$ to be meaningful.

IV. Integrals in Path Dependent Calculus

With the definition of the displacement derivative, eq. (3.5), we have come enormously far: We have explained non-commutivity of derivatives, come to understand the area and volume derivatives, made contact with differential geometry and the Papapetrou equations, and caught a glimpse of the curious non-associativity of surface dependent calculus. However, what eventually justifies the displacement derivative is not these things, but our ability to use it to create new and useful fundamental theorems. Thus, we devote the next few sections to defining integration and fundamental theorems for path dependent functions.

A. Path Integration

We define the path integral as a mapping of a path dependent function,

$$\int_{\underline{P_0}}^{P} : V \to V ; G(\underline{P}) \mapsto \int_{\underline{P_0}}^{P} G(\underline{P}') \ , \tag{4.1}$$

given by the Riemann sum,

$$\int_{\underline{P_0}}^{P} G(\underline{P}') \equiv \left(\frac{1}{\epsilon}\right) \lim_{NT(P)\to\infty} N^{-1} \sum_{n=0}^{NT(P)} G\left[\underline{P_0} \Rightarrow (P)_{n/N}\right] \ , \tag{4.2}$$

where $(P)_{n/N}$ is a partition of P as defined in eq. (3.3).

With this definition, we can form two distinct fundamental theorems. The first has a displacement derivative *inside* the path integral,

$$\int_{\underline{P_0}}^{P} \mathbf{d_v} G(\underline{P}'^{\star}) = G(\underline{P}_0 \Rightarrow P) - G(\underline{P}_0) \ , \tag{4.3}$$

where $\mathbf{v}$ is the tangent vector at the end of the path $\underline{P}'$. This is the direct generalization of our earlier path integral theorem, eq. (2.7). The second fundamental theorem, as we alluded to in eq. (2.10), has a derivative *outside* the path integral:

$$\mathbf{d_v} \int_{\underline{P_0}}^{P^{\star}} G(\underline{P}') = G(\underline{P}_0 \Rightarrow P^{\star}) \ . \tag{4.4}$$

This result is more than a coincidence; in fact, the definition of $\mathbf{d_v}$ eq. (3.5) was largely chosen so that this theorem would work out. So, in a sense, all of path dependent calculus has it's origin in eq. (4.4). Furthermore, eq. (4.4) encourages us to ponder the existence of higher dimensional theorems,

$$\mathbf{d}^2_{\mathbf{v}\wedge\mathbf{w}} \iint_{surface} H = H, \quad \mathbf{d}^3_{\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}} \iiint_{volume} I = I, \quad etc. \quad \ldots \ . \tag{4.5}$$

Though we don't yet fully understand the calculus of surface and volume dependence etc., we can use these equations to guide further generalization; just as eq. (4.4) guided the development of path dependent calculus.

B. Surface Integrals

The two fundamental theorems we have discussed so far, equations (4.3) and (4.4), both involve the displacement derivative. What about the area derivative? Does it have any fundamental theorem? In fact, it does, a fundamental theorem involving the *surface* integral of an area derivative. Before we can express this theorem, however, we have to define what it means to integrate a *path dependent* function over a surface.

We would like to think of a surface as a parameterized set of paths,

$$\underline{P} = \underline{P}(s); \quad s \in [0, S] \ , \tag{4.6}$$

where the paths all share the same endpoints and parameterization interval, T. Therefore, we can imagine a surface to be "sheet" swept out by a continuum of

paths bound between two points, $\underline{\mathbf{R}}_i$ and $\underline{\mathbf{R}}_f$. Alternatively, we could represent a surface as a set of points parameterized by two variables, (s,t):

$$\underline{\mathbf{R}} = \underline{\mathbf{R}}(s,t); \quad s \in [0,S],\ t \in [0,T] \ , \tag{4.7}$$

such that $\underline{\mathbf{R}}(s,0)$ and $\underline{\mathbf{R}}(s,T)$ are both fixed points in space, independent of s.

Having said this, we define the surface integral on a function, $K(\underline{P},\mathbf{v},\mathbf{w})$ which depends both on a path $\underline{P}$ and the *change in* $\underline{P}$ through two vectors, $(\mathbf{v},\mathbf{w})$, which can be visualized as tangent to the surface of change. Thus, the surface integral operator is a mapping,

$$\int\limits_{\underline{P}(s);s\in[0,S]}\!\!\int : V \rightarrow V; K(\underline{P},\mathbf{v},\mathbf{w}) \mapsto \int\limits_{\underline{P}(s);s\in[0,S]}\!\!\int K(\underline{P},\mathbf{v},\mathbf{w}) \ , \tag{4.8}$$

given by the double Riemann sum,

$$\int\limits_{\underline{P}(s);s\in[0,S]}\!\!\int K(\underline{P},\mathbf{v},\mathbf{w}) \equiv \left(\frac{1}{\epsilon^2}\right) \lim_{NT,NS\rightarrow\infty} N^{-2} \sum_{m=1}^{NS} \sum_{n=1}^{NT} K\left[\underline{P}(m,n),\mathbf{v}(m,n),\mathbf{w}(m,n)\right] \ , \tag{4.9a}$$

$$where: \quad \underline{P}(m,n) = \left(\underline{P}\,[m/N]\right)^{\star}_{n/N} \Rightarrow {}_{n/N}\left(\underline{P}\,[m/N]\right) \ , \tag{4.9b}$$

$$\mathbf{v}(m,n) = \left.\frac{d\underline{\mathbf{R}}(m/N,t)}{dt}\right|_{n/N} \ , \tag{4.9c}$$

$$\mathbf{w}(m,n) = \left.\frac{d\underline{\mathbf{R}}(s,n/N)}{ds}\right|_{m/N} \ . \tag{4.9d}$$

This equation looks formidable, but really the idea is quite simple: we are adding up K over all of the paths $\underline{P}(s)$ and over the changes which are occurring at each point along those paths. Combining the definition of the surface integral with the area derivative, equations (3.9) and (4.9), we discover the remarkable fundamental theorem,

$$\int\limits_{\underline{P}(s);s\in[0,S]}\!\!\int \mathbf{d}^2_{\mathbf{v}\wedge\mathbf{w}} G(\underline{P}') = G[\underline{P}(S)] - G[\underline{P}(0)] \ . \tag{4.10}$$

This result is immediately distinguished from other fundamental theorems by one extraordinary feature: *it integrates in a step of two*. This means, specifically, that it relates a *double* integral over a *second order* derivative to something with no integrals and no derivatives. By contrast, fundamental theorems in standard theory always integrate *once* over a *single* derivative. This is true of differential forms, whose fundamental theorem integrates over *only one* exterior derivative[3],

$$\int\limits_{\Sigma} \mathbf{d}\omega = \int\limits_{\partial\Sigma} \omega \ . \tag{4.11}$$

We would never see a fundamental theorem which integrates two exterior derivatives, because such a theorem would be trivial by the Poincare Lemma, $\mathbf{d}^2\omega = 0$. If we complete the generalization to surface and volume dependence etc., we expect to find even bigger fundamental theorems which integrate in k steps:

$$\iiint\limits_{X^3} \mathbf{d}^3_{\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}} H = \Delta H \, , \quad \iiiint\limits_{X^4} \mathbf{d}^4_{\mathbf{t}\wedge\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}} H = \Delta H \, , \tag{4.12ab}$$

$$\iint \cdots \int\limits_{X^k} \mathbf{d}^k_{\mathbf{x_1}\wedge\mathbf{x_2}\wedge\ldots\wedge\mathbf{x_k}} H = \Delta H \; . \tag{4.12c}$$

Eq.(4.12abc) is, doubtless, an impressive geometric hierarchy of fundamental theorems; but still, we need to do much more thinking before eq. (4.12) can be given any definite meaning. It is our great hope that this profound generalization may some day be realized.

We should mention one other remarkable property of the area derivative fundamental theorem: it automatically includes the 2-D Stoke's theorem as a special case, as we can verify by evaluating eq. (4.10) using a line integral as in eq. (3.8). It is significant that we can derive the 2-D Stoke's theorem from the area derivative theorem, eq. (4.10). In fact, perhaps *all* Stoke's theorems can be derived from suitably generalized fundamental theorems, as in eq. (4.12). In this view, Stoke's theorem, which was previously thought to be "fundamental", is no more than a shadow of more powerful theorems from a more general calculus.

V. Conclusion

A. Overlap with Standard Theories

1. Differential geometry

The author is aware that standard theory has already made contact with path dependent calculus, at least partially, through the notion of *curvature* in differential geometry. Indeed, this is why the ideas in this paper have anything to do with the Papapetrou equations for spinning particles in curved space.

We should emphasize, however, that path dependent calculus actually gives us more than differential geometry alone. In particular, differential geometry is concerned with only one type of path dependent function, a basis vector in curved space, while path dependent calculus concerns *all* path dependent functions (as long as it satisfies internal linearity). But also, differential geometry makes it difficult to appreciate the significance of *path dependence* as a powerful and logical extension of multivariable calculus. This situation is aggravated by the definition of the covariant derivative, ∇_σ, which allows one to calculate in curved space without any reference to path dependent basis vectors. Thus, with the path dependent objects out of sight, one sees no need for a calculus of path dependent functions.

2. The Calculus of Variations and Functional Analysis

When people first hear about this calculus, they often think that it must bear some close relationship to the calculus of variations and functional analysis. Actually, the relationship is not as strong as one might initially suppose, the two theories are capable of describing each other, but not in a simple or natural way.

Functional Analysis considers paths, surfaces, and volumes to be like *points* in an infinite dimensional vector space. Hence, in this view, a path dependent function is really no more than a function of *position* in infinite dimensions. This concept allows us to directly generalize the ideas of position dependent calculus to functions of path, surface, and volume etc. Thus, a directional derivative in infinite dimensions can be visualized as an arbitrary path variation which could, if specifically chosen, describe the displacement or area derivatives. However, this approach is not very pleasing, the significance of the displacement derivative is drowned out by a vast sea of other equally possible path variations which are of no particular importance. We could, on the other hand, approach this from the opposite direction, that is, describe a general path variation as a sum of displacement and area derivatives. However, it is not clear that we gain an appreciable amount for our efforts. Thus, while path dependent calculus could be made equivalent to Variational Analysis, and vice versa, it is not particularly natural or illuminating to do so.

3. The Geometric Calculus of Hestenes and Sobczyk

There may be some question as to how much the calculus of this paper has in common with the Geometric Calculus of Hestenes and Sobczyk, after all, both theories define differentiation with respect to vectors and bivectors etc., and both are, in some sense, "geometric". However, the two theories are essentially different in their domain of consideration. Geometric Calculus concerns, mostly, functions which depend on a general element of the Clifford Algebra, that is, a multivector[5]. Our calculus concerns functions of paths and surfaces etc., and makes only modest use of Clifford Algebra. Perhaps there is something to be gained by trying to unify these two theories, but for now, it is clear that they are not the same.

B. Where Do We Go From Here?

1. Completing the Generalization

In the author's opinion, the fate of this calculus depends crucially on our ability to generalize it to functions of arbitrary hyper-surfaces n-dimensions. Consistency of principle demands that, if we did it for paths, we must be able to do it for surfaces and volumes etc. However, the question is very complex, and the author has not yet been able to find a satisfactory generalization. If a general definition for the displacement derivative is ever found, we expect it to satisfy the following properties:

(i) It must be non-associative in a greater number of parentheses when it differentiates functions of more complex hyper-surfaces.

(ii) It must yield intuitive definitions for the area, volume etc. derivatives through relationships such as eq. (3.11) and eq. (3.20).

iii) It must reduce to the position and path dependent definitions of $\mathbf{d_v}$ as a special case.

iv) It must satisfy internal linearity for a sufficiently broad class of functions.

v) It must allow us to write path, surface, volume etc. integral fundamental theorems with the derivative on the *outside*, as in eq. (4.5).

vi) It must also allow us to write path, surface, volume etc. fundamental theorems with the derivative inside, as in eq. (4.12).

This is a lot of mathematical weight to be carried by a single definition, so we must define $\mathbf{d_v}$ with great care. The author is currently working on some ideas.

2. A New Approach to Differential Geometry

It is quite striking how easily we were able to derive fundamental results from differential geometry through a few modest insights and simple definitions. Concepts such as curvature, torsion, and the Bianchi identity which are usually deeply buried in formalism, are cleanly derivable with path dependent calculus. Perhaps differential geometry might benefit from a complete pedagogical re-derivation within the framework of path dependent calculus, a problem currently under investigation.

One of the interesting possibilities such a re-derivation might offer is the prospect of generalizing the concept of space to allow for *surface* dependent basis vectors. Thus, the Bianchi identities would no longer vanish and the volume derivative would give us a new fundamental tensor,

$$\frac{\partial^3 \hat{\mathbf{e}}_\mu}{\partial V^{\alpha\beta\gamma}} = U_{\alpha\beta\gamma\mu}{}^{\sigma}\, \hat{\mathbf{e}}_\sigma \ . \tag{5.1}$$

What this equation essentially means is that the concept of parallel transport of a point along a path is replaced by the concept of parallel transport of a *path through a surface*. Possible applications to String Theory come to mind, but at this point, the idea is far from being concretely realized.

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